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On the impulsively started rotating sphere

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The velocity field generated in a fluid of viscosity, ν, by impulsively starting at time $t = 0$, a sphere of radius a spinning with angular velocity $\mathbf{\Omega}$ about a diameter is described using a new expansion variable $2\sqrt{\nu t}/r$. It is first shown how the standard time-dependent boundary-layer equations can be modified to give series solutions satisfying *all* the boundary conditions. Next, that these new solutions are relevant when the Reynolds number $R = a^2\Omega/\nu$ goes to infinity in such a way that $R^{\frac{1}{3}}\Omega t$ is large. Lastly, solutions are given, applicable at small times for non-zero Reynolds numbers. These last expansions show that the velocity components decay algebraically rather than exponentially at large distances.

1. Introduction

The problem of the flow about a sphere rotating in a viscous fluid is a fascinating one which has numerous applications in engineering, astrophysics and meteorology. Attention, here, will be restricted to problems which are symmetric about a fixed axis of spin for reasons of simplicity. Even then the problem is fully three-dimensional. If r, θ, ϕ are spherical polar co-ordinates fixed in space with an origin at the centre of the sphere and u, v, w are the velocity components in these directions, respectively, the boundary condition of no-slip generates a circumferential velocity w and this in turn generates radial and transverse components u, v, via the centripetal acceleration terms in the equations of motion.

In the steady flow problem the Reynolds number $R = a^2\Omega/\nu$ where a is the radius of the sphere, Ω is the angular velocity and ν is the viscosity of the fluid, is the dimensionless parameter that determines the flow. In the non-steady problem the time scale, T provides a second parameter. This has usually been taken as the inviscid scale $T = \Omega^{-1}$. Most attention has been paid to the impulsive motion problems in which the Reynolds number is large as these are the simplest mathematically. Using the boundary-layer approximation series solutions have been given in powers of the time by Nigam & Rangasami (1953) for the spinning spheres and by Illingworth (1954) for the more involved problem of the spinning projectile. Illingworth's results include Nigam & Rangasami's as a special case. For the spinning sphere, the radial velocity component was found not to satisfy the boundary condition at large distances.

More recently, Benton (1965) choosing a viscous time scale $T = \nu/a^2$ has given solutions based on the assumption that the dominant velocity component was that in the azimuthal direction. He found an approximate solution for the secondary flow and was able to satisfy all the boundary conditions at large distances. He did not make at all clear the ranges of Reynolds number and time over which his solutions were of value.

The method of this paper is to choose the viscous time as Benton did and then to express the solutions in power series in a variable which is a combination of the time and the radial co-ordinate, viz. $s = 2\sqrt{\nu t}/r$. The expansions obtained include the boundary-layer results and Benton's results as special cases and satisfy the complete Navier–Stokes equations and all the boundary conditions. The precise regions and ranges of values of t and R in which the previously known solutions are of value are also clarified.

2. Equations and boundary conditions

Consider a sphere of radius a at rest in a homogeneous incompressible viscous fluid of density ρ and kinematic viscosity ν. Suppose that at time $t = 0$ the sphere is given an instantaneous angular velocity $\mathbf{\Omega}$ about a diameter and that this is maintained at all subsequent times. The equations governing the motion are the Navier–Stokes equations

$$\partial \mathbf{v}/\partial t + (\mathbf{v} . \nabla)\,\mathbf{v} = -\rho^{-1}\,\mathrm{grad}\,p + \nu\nabla^2\mathbf{v}, \tag{1}$$

$$\mathrm{div}\,\mathbf{v} = 0. \tag{2}$$

In these, $\mathbf{v}$ and p are the velocity and pressure. The initial conditions, and the boundary conditions, of no slip at the surface and vanishing velocity at infinity imply that:

$$\left.\begin{array}{lll} \text{at} & t = 0, \quad \mathbf{v} = 0 \quad \text{for all} \quad r > a, \\ \text{for} & t \geqslant 0, \quad u = v = 0, \quad w = a\Omega\sin\theta \quad \text{for} \quad r = a, \\ \text{and} & \mathbf{v} \to 0 \quad \text{as} \quad r \to \infty. \end{array}\right\} \tag{3}$$

Non-dimensional variables are now introduced according to the scheme

$$t = Tt', \quad \mathbf{v} = a\Omega\mathbf{v}', \quad \mathbf{r} = a\mathbf{r}', \quad p = \rho a^2\Omega^2 p', \tag{4}$$

where T is a typical time which is left unspecified for the time being. Equations (1) and (2) using (4) become

$$(T\Omega)^{-1}\partial\mathbf{v}/\partial t + (\mathbf{v} . \nabla)\,\mathbf{v} = -\,\mathrm{grad}\,p + R^{-1}\nabla^2\mathbf{v}, \tag{5}$$

$$\mathrm{div}\,\mathbf{v} = 0. \tag{6}$$

where the primes have been omitted. Equations (7) and (8) show the forms of the transverse and azimuthal components of the momentum equation. There is a balance between the time dependent terms, the quadratic convection terms, the pressure, and the linear viscous terms

$$\frac{v_t}{\Omega T} + \left(\text{quadratic in } u \text{ and } v \; - \frac{w^2\cot\theta}{r}\right) = \frac{p_\theta}{r} + \frac{1}{R}(v_{rr} + \text{linear in } u \text{ and } v), \tag{7}$$

$$\frac{w_t}{\Omega T} + (\text{quadratic in } u \text{ and } v \text{ or } v \text{ and } w) = \frac{1}{R}(w_{rr} + \text{linear in } w). \tag{8}$$

The full equations appropriate to an axially symmetric problem are given in Benton's paper.

3. Previous investigations

Immediately after the impulsive start, the velocity field consists of a vortex sheet at the surface of the sphere. Outside this sheet the velocity is zero. At small times diffusion dominates convection, and the fluid is moving almost entirely in the azimuthal direction. The two methods that have been found for solving the problem at small times depend on the relative importance of these two effects.

If times are considered for which diffusion dominates convection, then disturbances are confined to a region whose thickness is proportional to

$$(\nu t T)^{\frac{1}{2}} = a(t\,.\,T\Omega/R)^{\frac{1}{2}}.$$

If R is large this region will still be narrow when $T = O(\Omega^{-1})$, i.e. there will be no boundary layer of thickness $O(R^{-\frac{1}{2}})$ on the body. Under such conditions, it is permissible to write

$$T = \Omega^{-1}, \quad r = 1 + R^{-\frac{1}{2}} Z^*, \quad u = R^{-\frac{1}{2}} u^*. \tag{9}$$

The equations for the boundary-layer region are found by substituting (9) into the equations of motion, and by taking the limit $R \to \infty$ assuming the starred variables are $O(1)$. A typical boundary-layer equation has the form

$$\underline{w_t} + (\text{quadratic in } u^* \text{ and } v \text{ or } v \text{ and } w) = \underline{w_{Z^*Z^*}}. \tag{10}$$

The complete set is an order lower than the Navier–Stokes equations so one boundary condition has to be discarded. Expansions in powers of t can be found by iteration, one of the first-order equations being given by the terms underlined in (10). Illingworth gave the solutions as (in dimensional variables)

$$\left.\begin{aligned} u &= -16\pi^{\frac{3}{2}}\Omega^2\nu^{\frac{1}{2}}t^{\frac{3}{2}}(3\cos^2\theta^{-1})\{G(\eta) + O(\Omega^2t^2)\}, \\ v &= 8\pi^{\frac{3}{2}}a\Omega^2 t\sin\theta\cos\theta\{G'(\eta) + O(\Omega^2t^2)\}, \\ w &= a\Omega\sin\theta\cos\theta\{w_0(\eta) + [\sin^2\theta\, w_{11}(\eta) + \cos^2\theta\, w_{12}(\eta)]\,\Omega^2t^2 + O(\Omega^4t^4)\}, \end{aligned}\right\} \tag{11}$$

where $\eta = (r-a)/(2\sqrt{\nu t})$. As $\eta \to \infty$, $G(\eta)$ tends to a constant so that the boundary condition on the normal velocity component is not satisfied at large distances.

When the azimuthal component of velocity is much larger than the other components and diffusion and convection are comparable

$$u, v \ll w \quad \text{and} \quad T = \Omega^{-1}R. \tag{12}$$

With these assumptions Benton argued as follows. In the early stages the primary flow w, dominates the secondary flow components u, v so at first the non-linear terms in (8) may be neglected. For similar reasons the terms in the transverse and radial equations which are quadratic in u and v may be omitted; the terms of $O(w^2)$ must be retained in full. He then went on to solve exactly the linear equation for w and showed that w was nearly similar in time provided $\tau = \Omega t/R < 10^{-3}$, the similarity variable being $\eta_b = (r-a)/[\pi^{\frac{1}{2}}a(1-e^{\tau}\operatorname{erfc}\tau^{\frac{1}{2}})]$. The approximation $w = \operatorname{erf}\eta_b$ was then used in the partially linearized radial and transverse equations to find solutions for u and v. He found

$$\left.\begin{aligned} u &= -2a^7\Omega^2\nu^{-1}d^3r^{-4}(3\cos^2\theta - 1)\{G(\eta_b) + O(t^2)\}, \\ v &= 2a^7\Omega^2\nu^{-1}d^2r^{-1}\sin\theta\cos\theta\{[r^{-2}G(\eta_b)]_r + O(t^2)\}, \\ w &= a\Omega\sin\theta\{W(r,t) + O(t^2)\}, \end{aligned}\right\} \tag{13}$$

where $d = 1-e^{\tau}\operatorname{erfc}\tau^{\frac{1}{2}}$. As $t\to 0$ and $r\to a$ Benton's solutions reduce to the first terms in Illingworth's series. The solutions u, v, $w\to 0$ as $r\to\infty$, u showing an algebraic decay and v and w exponential decay, so in this sense are an improvement over the boundary-layer results. The other results of importance are, first, that v changes sign at a distance of about $1\frac{1}{2}$ radii from the centre of the sphere as required by continuity (though this could be a consequence of the approximate method used as in this region the errors introduced by taking W to be similar in time are large), and, secondly, that the profiles W could, by an appropriate choice of the similarity variable, be approximately related to the steady-state solutions of Howarth (1951).

4. Present method

The method described below is capable of including both these results in terms of a single expansion procedure. A variable which is appropriate to the description of the developing boundary layer is that of Blasius, $\eta = (r-a)/[2\sqrt{(\nu t)}]$. This variable has been used in a variety of problems of the impulsive motion type and, in particular, in the now classical problem of Goldstein & Rosenhead (1936) on the motion of a circular cylinder perpendicular to its axis. There the radial velocity component to a first approximation was found to have the form

$$u \sim -2\sqrt{\nu t}(\eta-\pi^{-\frac{1}{2}})\cos\theta \sim -(r-a)\cos\theta,$$

whereas the potential flow over the body has the behaviour

$$u \sim -(1-a^2r^{-2})\cos\theta.$$

The author felt that if expansions in $2(\nu t)^{\frac{1}{2}}/r$ were developed then these should be equivalent to the boundary-layer results near the surface and should also possess the correct asymptotic forms. (For small values of t, Benton's solutions for u and v depend on $(\nu tr^{-2})^{\frac{3}{2}}$.) In the cylinder problem it was found that expansions could be obtained in powers of $2(\nu t)^{\frac{1}{2}}/r$ which reduced to the boundary-layer results in the limit $R\to\infty$ for small r and which differed at large values of r from the exact potential flow by terms of $O(1/R)$.

However, an infinite set of arbitrary functions appeared in these terms of $O(1/R)$ which could not be determined in any obvious way. For the spinning sphere, this last difficulty does not arise. In the boundary-layer solutions of Nigam and Rangasami, and of Illingworth, the equations are simplified at the outset and the boundary condition on u as $r\to\infty$ is not satisfied. The choice of $2(\nu t)^{\frac{1}{2}}/r$ as the expansion variable enables solutions to be written down which satisfy the complete equations and which do satisfy all the boundary conditions.

A dimensionless stream function ψ, is introduced at this stage, the complete transformation being

$$\left.\begin{aligned} r^2\sin\theta u = a^3\Omega\psi_\theta, \quad r\sin\theta v = -a^3\Omega\psi_r,\\ \eta = (r-a)/2(\nu t)^{\frac{1}{2}}, \quad s = 2(\nu t)^{\frac{1}{2}}/r.\end{aligned}\right\} \tag{14}$$

Equation (1), using (14), gives

$$\left(\frac{1-\eta s}{s}\right)^3 \frac{1}{R} D^2 \left\{D^2 - 2s\frac{\partial}{\partial s} + 2\eta\frac{\partial}{\partial \eta}\right\}\psi$$
$$= 2w\left\{\cos\theta\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)w - s\sin\theta w_\theta\right\}$$
$$-(s^2\sin\theta)^{-1}(1-\eta s)^2\left\{\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)\psi(D^2\psi)_\theta - \psi_\theta\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)D^2\psi\right\}$$
$$+2(s\sin\theta)^{-2}(1-\eta s)^2\left\{\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)\psi\cos\theta - s\sin\theta\psi_\theta\right\}D^2\psi, \tag{15}$$

and

$$\left\{\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)^2 + 2s\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right) + \frac{s^2}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin\theta}\right) - 2s\frac{\partial}{\partial s} + 2\eta\frac{\partial}{\partial\eta}\right\}w$$
$$= (\sin\theta)^{-1}Rs(1-\eta s)\left\{\psi_\theta\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s} + s\right)w - \left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)\psi\left(\frac{\partial}{\partial\theta} + \cot\theta\right)w\right\}, \tag{16}$$

where
$$D^2\psi = \left\{\left(\frac{\partial}{\partial\eta} - s^2\frac{\partial}{\partial s}\right)^2 + s^2\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}\right)\right\}\psi. \tag{17}$$

The equation of continuity is satisfied automatically.

These possess solutions in powers of s, in which the Reynolds number enters in a particularly simple manner. The solutions may be written as

$$\psi = s^3R\sum_{m,n=0}^{\infty} s^m(R^2s^4)^n\psi^{(mn)}(\eta,\theta), \tag{18}$$

$$w = \sum_{m,n=0}^{\infty} s^m(R^2s^4)^n w^{(mn)}(\eta,\theta). \tag{19}$$

The terms with $m = 0$ correspond to the boundary-layer solutions of Nigam & Rangasami and Illingworth. The functions $\psi^{(0n)}$, $w^{(0n)}$ can be deduced from those of the standard boundary-layer analysis. Illingworth's results can be used to determine $\psi^{(00)}$, $w^{(00)}$, and the value of $[(\partial/\partial\eta)w^{(01)}]_{\eta=0}$. The neglect of the terms for which $m \neq 0$ requires that

$$R^2s^4 \gg s \quad\text{and}\quad s \ll 1, \tag{20}$$

i.e. $$R^{\frac{1}{3}}\,\Omega ta^2/r^2 \gg 1, \quad \Omega ta^2/Rr^2 \ll 1.$$

For a given value of Ωt and a large value of R the approximations break down when $\Omega tR^{\frac{1}{3}} \gg r^2/a^2$. The modified boundary-layer series may be regarded as the solution of the equations when $R\to\infty$, $\Omega t\to 0$, in such a way that $\Omega tR^{\frac{1}{3}}$ is large. It is valid within a sphere of radius $O([\Omega tR^{\frac{1}{3}}]^{\frac{1}{2}})$. In principle, the next series of terms with $m = 1$ could be computed having determined the complete modified boundary-layer series. This would give a correction of $O([\Omega t/R]^{\frac{1}{2}})$ to the initial series.

Benton's solutions correspond to the terms with $n = 0$. In fact his exact solution for the primary flow, W, is given by

$$W = \sum_{n=0}^{\infty} s^m w^{(m0)}(\eta,\theta), \tag{21}$$

and his approximations to the secondary flow are approximations to the series

$$\psi = s^3 R \sum_{n=0}^{\infty} s^m \psi^{(m0)}(\eta, \theta). \tag{22}$$

Benton's solution requires that

$$R^2 s^4 \ll s,$$

i.e.

$$R^{\frac{1}{3}}\, \Omega t a^2/r^2 \ll 1, \tag{23}$$

and so provides a solution for a given Reynolds number which is valid at small times and at large distances.

If the functions $\psi^{(mn)}$ and $w^{(mn)}$ are regarded as the elements of infinite rectangular arrays, then the modified boundary-layer series and successive corrections to it corresponds to summing by rows whereas Benton's method and successive corrections to it corresponds to summing by columns. When $R = 10^5$ and $\Omega t > 0{\cdot}005$ the second term in the boundary-layer expansion dominates the second term in Benton's expansion, and when $R = 10^5$ and $\Omega t > 1$ the third term in the boundary-layer expansion dominates the second term in Benton's expansion.

5. Boundary-layer expansion

We now develop a solution which includes the boundary-layer expansion but which is valid for a larger range of r. Since the neglected terms in the series decay algebraically rather than exponentially for a given Reynolds number there will be an upper limit to the value of r for which the solution will be applicable.

When $R^2 s^3 \gg 1$ and $s \ll 1$ the series (18) implies that at large distances from the surface

$$\psi = 8r^{-3}(\nu t)^{\frac{3}{2}} R \sum_{n=0}^{\infty} (4R\nu t/r^2)^{2n} \psi^{(0n)}(\eta, \theta) \sim \sum_{n=0}^{\infty} O\left(\frac{1}{r^{4n+3}}\right) \psi^{(0n)}.$$

In order that the boundary conditions on the normal velocity component may be satisfied it is sufficient to demand that $\psi^{(0n)}/\eta^{4n+3} \to 0$ as $\eta \to \infty$. The functions $\psi^{(0n)}$ are those that arise in fact in a standard boundary-layer analysis and those that are known satisfy the stronger conditions $\psi^{(0n)}/\eta \to 0$. As the terms of (15)–(17) that generate (18) and (19) are

$$w_{\eta\eta} + 2\eta w_\eta - 2s w_s = (\sin\theta)^{-1} R s (w_\eta \psi_\theta - w_\theta \psi_\eta), \tag{24}$$

$$(\psi_{\eta\eta} + 2\eta\psi_\eta - 2s\psi_s)_{\eta\eta} = R s^3 \left\{\cos\theta w^2 + s^{-2}\left(\frac{\psi_\theta \psi_{\eta\eta}}{\sin\theta} - \frac{\psi_\eta}{\sin^2\theta}(\sin\theta_{\psi_\eta})_\theta\right)\right\}_\eta, \tag{25}$$

it is suggested that these equations should be taken as the boundary layer equations applicable at small times. Equation (25) has the first integral

$$(\psi_{\eta\eta} + 2\eta\psi_\eta - 2s\psi_s)_\eta = R s^3 \left\{\cos\theta w^2 + s^{-2}\left(\frac{\psi_\theta \psi_{\eta\eta}}{\sin\theta} - \frac{\psi_\eta}{\sin^2\theta}(\sin\theta\psi_\eta)_\theta\right)\right\}, \tag{26}$$

as $\psi_\eta \to 0$ as $\eta \to \infty$. Equations (24) and (26) reduce to the boundary-layer equations when s is replaced by $2(\nu t)^{\frac{1}{2}}/a$ in them. This minor modification removes the difficulty of the asymptotic form of the stream function and gives the boundary-layer solution as $r \to a$; in particular the value of the skin friction and the results on radial outflow are unaltered.

The solution of (24) and (26) proceeds iteratively. The solutions for $w^{(0n)}$ and $\psi^{(0n)}$ can be expressed as sums of products of functions of x and functions of η. Thus

$$\left.\begin{aligned} w^{(00)} &= Uf^{(0)}, \\ \psi^{(00)} &= U^2 U_\theta s^{(0)}, \\ w^{(01)} &= U^3 f^{(11)} + U U_\theta^2 f^{(12)}, \end{aligned}\right\} \tag{27}$$

where $U = \sin\theta$ and the functions $f^{(0)}, s^{(0)}, f^{(11)}$ and $f^{(12)}$ satisfy ordinary differential equations deducible from those appearing in Illingworth's paper. The solutions for $f^{(0)}$ and $s^{(0)}$ can be readily found. Numerical solutions for $f^{(11)}$ and $f^{(12)}$ were also obtained.

6. Expansions in *s*

For a finite Reynolds number the terms of $O(s)$ in the expansions of w and $\psi/(s^3 R)$ dominate the first-order boundary-layer perturbations at small times. The lower-order terms in the series for w are found by iterating with the homogeneous form of (16). They satisfy the differential equations

$$\left.\begin{aligned} F_0(w^{(00)}) &= 0, \\ F_1(w^{(10)}) &= -2w_\eta^{(00)}, \\ F_2(w^{(20)}) &= -(w_{\theta\theta}^{(00)} + \cot\theta\, w_\theta^{(00)} - (\sin\theta)^{-2} w^{(00)}), \end{aligned}\right\} \tag{28}$$

where F_n is the differential operator defined by $F_n(f) = f_{\eta\eta} + 2\eta f_2 - 2nf$. Solutions of these are readily obtained as

$$\left.\begin{aligned} w^{(0)} &= Ur_0 &&= Uf^{(0)}, \\ w^{(1)} &= U(+\tfrac{1}{2}r_{-1} - r_1) &&= Uf^{(1)}, \\ w^{(2)} &= U(-\tfrac{1}{2}r_0 + 2r_2) &&= Uf^{(2)}, \end{aligned}\right\} \tag{29}$$

which all decay exponentially as $\eta \to \infty$. The functions r_n are defined by the relations $r_0 = \operatorname{erfc}\eta$, $r_n = (r_{n+1})_\eta$, $r_n(\infty) = 0$.

Benton computed the primary flow ignoring the influence of the secondary flow, as here by ignoring the right-hand side of (16) and gave the exact solution for the homogeneous equations satisfying the boundary conditions (3) as

$$w = U\frac{a^2}{r^2}\left\{\operatorname{erfc}\left(\frac{r-a}{2(\nu t)^{\frac{1}{2}}}\right) + \frac{r-a}{a}\exp\left(\frac{r-a}{a} + \frac{(\nu t)^{\frac{1}{2}}}{a}\right)\operatorname{erfc}\left(\frac{r-a}{2(\nu t)^{\frac{1}{2}}} + \frac{(\nu t)^{\frac{1}{2}}}{a}\right)\right\}. \tag{30}$$

Equations (29) are just the first three terms of the expansion of (30) in powers of s.

The lowest-order terms in s in the series for the stream function arise from the terms on the right-hand side of (15) that are quadratic in the azimuthal velocity component. The trigonometrical dependence can be removed by writing

$$\psi = U^2 U_\theta \overline{\psi}, \quad w = U\overline{w}. \tag{31}$$

Equation (10) can then be written as

$$(1-\eta s)^{-2}\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}\right)\left\{(1-\eta s)^2\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}+2s\right)\right.$$
$$\left.\times\left[\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}\right)^2-6s^2+2\eta\frac{\partial}{\partial\eta}-2s\frac{\partial}{\partial s}\right]\right\}\overline{\psi}$$
$$=(1-\eta s)^{-s}Rs^3\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}\right)\{(1-\eta s)^2\overline{w}^2\}, \tag{32}$$

which has the first integral

$$\left\{\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}+s\right)\left(\frac{\partial}{\partial\eta}-s^2\frac{\partial}{\partial s}\right)^2-6s^2+2\eta\frac{\partial}{\partial\eta}-2s\frac{\partial}{\partial s}\right\}\overline{\psi}=(1-\eta s)^{-3}Rs^3\overline{w}^2, \tag{33}$$

the constant of integration being zero.

The first three terms in the expansion of ψ are such that

$$\psi^{(i)}=U^2U_\theta s^{(i)}, \tag{34}$$

where

$$F_2(s_\eta^{(0)})=f^{(0)2}, \tag{35}$$
$$F_3(s_\eta^{(1)})=F_3(s^{(0)})+6s_{\eta\eta}^{(0)}+3\eta f^{(0)^2}+2f^{(0)}f^{(1)}, \tag{36}$$
$$F_4(s_\eta^{(2)})=2F_4(s^{(1)})+8s_{\eta\eta}^{(1)}-18s_\eta^{(0)}+6\eta^2f^{(0)^2}+12\eta f^{(0)}f^{(1)}+2f^{(0)}f^{(2)}+f^{(1)^2}. \tag{37}$$

Equation (35) is identical with one which appears in the boundary-layer analysis. To determine the solution of (36) and (37) is a relatively easy but tedious process so it was preferred to integrate the equations numerically. In order to proceed the asymptotic forms of the solutions are required. As $\eta\to\infty$, $s^{(0)}\to c^{(0)}$, so from (35) to (37)

$$\left.\begin{aligned}&s^{(1)}\to c^{(0)}\eta+c^{(1)},\\&s^{(2)}\to c^{(0)}\eta^2+2c^{(1)}\eta+c^{(2)},\\&s^{(3)}\to c^{(0)}\eta^3+3c^{(1)}\eta^2+3c^{(2)}\eta+c^{(3)},\end{aligned}\right\} \tag{38}$$

where $c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}$ are constants. These suggest that at large distances from the surface, the solution of (33) takes the form

$$\psi=s^3R\left\{\frac{c^{(0)}}{1-\eta s}+\frac{c^{(1)}}{(1-\eta s)^2}+\frac{c^{(2)}}{(1-\eta s)^3}+\dots\right\}$$
$$=\frac{4\nu tR}{r^2}\left\{\frac{2(\nu t)^{\frac{1}{2}}}{a}c^{(0)}+\frac{4\nu t}{a^2}c^{(1)}+\frac{8(\nu t)^{\frac{3}{2}}}{a^3}c^{(2)}+\dots\right\}.$$

This shows the dependence on r as Benton's solution.

7. Discussion

The ranges of validity of the two expansions can now be clarified. The aximuthal velocity component has the form

$$w=U\{f^{(0)}+sf^{(1)}+s^2f^{(2)}+s^3f^{(3)}+s^4f^{(4)}+O(s^5)+(U^2f^{(11)}+U_\theta^2f^{(12)})R^2s^4+O(R^4s^8)\}.$$

The f's satisfy the relationships

$$|f^{(i)}|<0{\cdot}10\quad(i=1,2),$$
$$|f^{(i)}|<0{\cdot}01\quad(i=1,2),$$

so convergence can be expected for values of s up to 0·1 and for values of Rs^2 up to 1. Now s and Rs^2 have their maximum values at the surface so these values imply that the maxima for $2\Omega t/R$ and $4\Omega t$ should be 0·01 and 1 respectively. The terms $O(R^2s^4)$ and $O(s)$ are comparable when

$$0{\cdot}01R^2s^4 = O(0{\cdot}1s),$$

i.e. $$\Omega t = O(R^{-\frac{1}{3}}).$$

Figure 1 shows the three curves $\Omega t = R/200$, $\Omega t = \frac{1}{4}$ and $\Omega t = R^{\frac{1}{3}}$. In region 1, the boundary solutions apply, in region 2 Benton's solution is adequately represented by the first few terms of the series in s only, in region 3 the series in s only

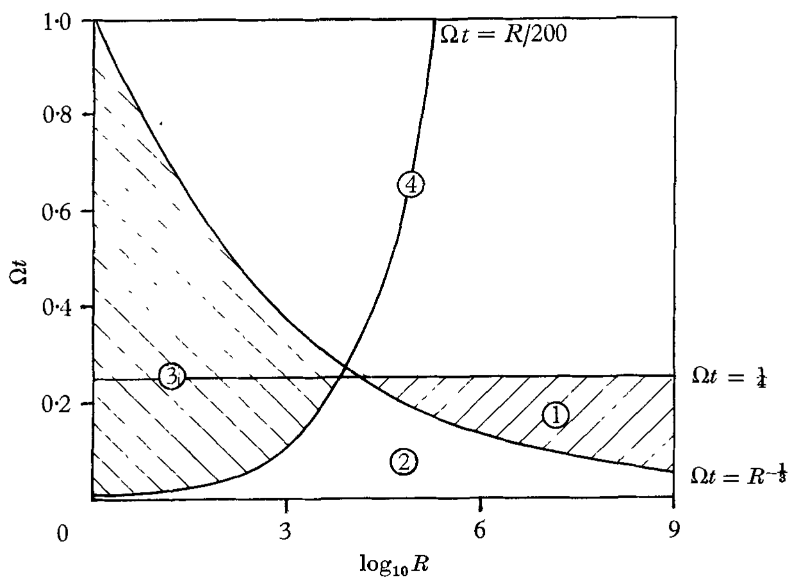

FIGURE 1. Approximate ranges of validity of boundary-layer solution, Benton's solution, and 's' series.

is slowly convergent to Benton's exact solution, and in region 4 the boundary-layer terms are important and the boundary-layer expansion is slowly convergent. By considering the form of the secondary flow, Benton concluded that his analysis was only valid if $\Omega t/R < 10^{-3}$. This is not sufficient. It is also necessary to impose the condition $\Omega t R^{\frac{1}{3}} < 1$.

The other main conclusion is that by modifying the boundary-layer expansion in a very simple way by replacing Ωt by $\frac{1}{4}Rs^2 = \Omega t(a/r)^2$ all the boundary conditions can be satisfied, the resulting expansions being valid for $R > 10^4$ and over a wider range of values of r/a than has previously been thought.

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